EE 359 Final Project Report
Literature Survey on Capacity Results for Wireless Channels

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1 Introduction

Wireless communication has become very important and ubiquitous. Channel capacity is one of the most important aspects of any communication system design and knowledge of the channel capacity for a particular environment informs us about the limits on achievable data rates irrespective of the available technology. This field saw a tremendous growth after Claude Shannon's 1948 seminal paper[2] titled ' Mathematical Theory of Communication ' which presents the mathematical principles for capacity analysis. Since then many capacity results have been derived for different situations like different channel models, single and multiple user systems,etc. Therefore, it is important to know the capacity results which have already been derived and thus taking inspiration from the derived results focus on capacity calculation problems which haven't been solved yet. This report presents single user wireless channel capacity results for many different situations - static and time varying channels, channels with and without CSI available at the receiver and transmitter and systems with single and multiple antennas at the input and output.

2 Overview

The report starts with a brief introduction to channel capacity, factors which influence it and some important results from Shannon's paper[2]. The following section then briefly describes channel modeling using the scattering function and important parameters like coherence bandwidth, etc. Three of the popular statistical models for also presented. The next section provides capacity results for SISO system - static channel, fading channel with CSI available only at the receiver, channel with CSI available at both the transmitter and receiver, noiseless but delayed CSI at the transmitter. The following section presents different notions of capacity - ergodic capacity, capacity with outage and delay limited capacity. Next section provides a brief comparison between the different capacities. The following section provides an introduction to MIMO systems and presents capacity results for it in different situations - static channel, fading channel with and without CSI at the receiver and transmitter. Finally a brief comparison between SISO and MIMO systems is made followed by the conclusion section.

3 Channel Capacity

Channel capacity refers to the maximum data rate at which information can be sent over a channel with arbitrary small error probability. Shannon in his seminal paper[2] provided fundamental theorems on channel capacity. Qualitatively some of the important results were :

1. It is possible to send information reliably even through an unreliable channel with arbitrary small error probability and he also provided the channel capacity formula for AWGN channel.
2. If we send information at a rate greater than the channel capacity, the probability of error will be bounded away from zero.
3. Codes exist which can achieve the channel capacity, however they may be very long or complex . He did not provide those codes but hinted at how to achieve them.

The results he provided are for a AWGN channel. However, practical channels introduce various other effects like fading, shadowing, etc in addition to additive noise. Moreover, information about the channel (CSI) may be available or unavailable to the receiver and transmitter. Broadly, channel capacity is influenced by the following parameters:

1. Type of environment, fading characteristics like Rayleigh, Rician fading, etc.
2. Availability of channel state information at the reciever and transmitter.
3. Use of multiple antennas at the reciever or transmitter
4. Single user versus multi user systems

The next section briefly describes various channel models.

4 Channel Models

As a wireless signal propagates through a channel it experiences various effects like attenuation, diffraction, mutiple reflections, etc. Moreover, the channel varies with time and therefore, the received signal strength varies with time. Therefore, a fading channel is characterised by a time varying impulse response c(τ,t). Since c(τ,t) is non - deterministic due to randomness in the channel, we work with its autocorrelation function $A_c(\tau,\Delta\text{t})$ and scattering function $S_c(\tau, \rho)$ defined as the fourier transform of $A_c(\tau, \Delta t)$ with respect to the Δt parameter. The delay power spectrum $S_c(\tau)$ of the channel is obtained by averaging $S_c(\tau, \rho)$ over ρ. The support of $S_c(\tau)$ gives us the multipath spread Tm of the channel and we define coherence bandwidth B_c as $\frac{1}{Tm}$. The Doppler power spectrum $S_c(\rho)$ is obtained by averaging $S_c(\tau, \rho)$ over τ. The support of $S_c(\rho)$ gives us the doppler spread B_d of the channel and we define coherence time T_{coh} as $\frac{1}{B_d}$.

A slowly fading channel has a large coherence time whereas a fast fading channel has a small coherence time. Also, measurments of the received signal separated in time greater than the coherence time are independent of each other. Similarly, received signals with frequencies separated by more than the coherence bandwidth are independent of each other. Channels can be broadly classified as frequency selective and frequency non-selective channels. If the signal bandwidth is more than the coherence bandwidth it is called a frequency selective channel or wideband channel since different frequencies undergo different attenuations. If the signal bandwidth is much less than the coherence bandwidth then we refer to the channel as frequency non-selective or flat fading channel.

4.1 Statistical Models for Narrowband Fading Channels

Statistical models for narrowband fading have been derived either from experiemental or from analytical considerations. These models characterize the probability distribution of the received signal amplitude. Three of the popular distributions are given below :

$$p(z) = \frac{z}{\sigma^2} \exp -\frac{z^2}{2\sigma^2} \qquad Rayleigh Distribution$$

z denotes the amplitude of the received signal, σ^2 is the average signal power.

$$p(z) = \frac{z}{\sigma^2} \exp -\frac{z^2 + s^2}{2\sigma^2} I_0(\frac{zs}{\sigma^2}) \qquad Rician Distribution$$

$2\sigma^2$ is the average power in the non-line of sight multipath components and s^2 is the power in the LOS component. I_0 is the modified Bessel function of zeroth order.

$$p(z) = \frac{2}{\Gamma(m)} (\frac{m}{\Omega})^m r^{2m-1} \exp(-\frac{mr^2}{\Omega}) \qquad Nakagami - m Distribution$$

Ω is the average received power, m is called the fading figure and can be selected to fit the measured data and $\Gamma()$ refers to the gamma function. Channel capacity will depend on the distribution selected for modeling the environment. Next section presents SISO system capacity results for static and time-varying environments and effects of CSI.

5 Capacity results for Single Input Single Output systems

This section presents capacity results for static and time varying channels having a probability distribution $p(\gamma)$ on the SNR γ. It also discusses capacities in the presence and absence of CSI at the receiver and transmitter.

5.1 Static Channel

The channel output y[i] at time i of a discrete time system can be represented as follows:

$$y[i] = a[i]x[i] + n[i]$$

Here n[i] is AWGN and a[i] causes input x[i] to attenuate due to path loss,etc. However, since this is a static channel a[i] remains constant with time. The capacity of this system is given by Shannon's famous formula :

$$C = Blog_2(1 + \gamma)$$

Here C is the capacity in bits per second, B is the signal bandwidth in Hertz and γ equals signal power by noise power. In the rest of the report C,B and γ will retain these meanings unless stated otherwise. In fading channels γ will refer to the instantaneous SNR. Shannon showed that this capacity result for a discrete memoryless time-invariant channel is obtained by maximising the mutual information I(X;Y) defined as follows :

$$I(X;Y) = \Sigma p(x,y) log(\frac{p(x,y)}{p(x)p(y)})$$

Here the summation is over all possible input x and output y values. Following subsections present capacity results for fading channels.

5.2 Channel Capacity with Receiver CSI only

We assume perfect channel state information is available at the receiver but no state information is available at the transmitter. The capacity in this case is given as follows:

$$C = \int_0^\infty B p(\gamma) log_2(1+\gamma) \mathrm{d}\gamma$$

$p(\gamma)$ depends on the fading distribution. Since the transmitter has no channel state information it transmits at a constant rate irrespective of the instantaneous SNR. The particular coding scheme which achieves this capacity will depend on the dynamics of the fading process. However, it must be long enough to reflect the ergodic nature of the fading process i.e $T_{code} \gg T_{coh}$

5.3 Channel Capacity with perfect Receiver and Transmitter CSI

When perfect channel state information is available at the transmitter and the receiver, power and rate adaption can be performed to maximise capacity. The optimal policy in this case is given by the water filling formula:

$$\frac{P(\gamma)}{P_{avg}} = \begin{cases} \frac{1}{\gamma_0} - \frac{1}{\gamma} & \gamma \gg \gamma_0 \\ 0 & otherwise \end{cases}$$

γ_0 depends on $p(\gamma)$ and can be obtained from the average power constraint. The corresponding optimal capacity is given as follows:

$$C = \int_{\gamma_0}^\infty B log_2(\frac{\gamma}{\gamma_0}) p(\gamma) \mathrm{d}\gamma$$

In the water filling policy both the rate and power are being adapted. However if fixed rate with only power adaptation is required, channel inversion policy given as follows can be used:

$$\frac{P(\gamma)}{P_{avg}} = \frac{\sigma}{\gamma}$$

$$\sigma = \frac{1}{E(\frac{1}{\gamma})}$$

$$C = B log_2(1+\sigma)$$

Channel inversion policy may not be suitable depending on the fading distribution, eg in Rayleigh fading channel inversion policy gives zero capacity. In this case, truncated channel inversion policy can be used where the channel is used only when the instantaneous SNR is above a given threshold.

5.4 Ideal CSI at Receiver only and Noiseless Delayed Feedback at the Transmitter

In this case CSI is available to the transmitter after some time delay. As mentioned by Ezio et al[3], "several examples have been worked out but no full solution has been found, as an elegant analytical solution for optimal power control does not seem to exist".

5.5 Unavailable Channel State Information

It is difficult to find the channel capacity in this case except for certain cases like finite state Markov channels and discrete Rayleigh fading channels.

5.6 Wideband Fading Channels

The results presented above are for a flat fading channel. In the case of a wideband fading channel where signal bandwidth is larger than the coherence bandwidth, capacity can be obtained by dividing the signal into smaller bandwidth blocks($B_{coh} \gg B_{block}$) and adding the capacities of the individual blocks obtained as in a flat fading channel.

6 Different Notions of Capacity

6.1 Ergodic Capacity

Ergodic capacity is obtained by integrating the instantaneous capacity $Blog_2(1+\gamma)$ over the fading distribution $p(\gamma)$. The basic assumption is that transmission time is greater than T_{coh} so that the long term ergodic properties of the fading channel are realised. Therefore, the code used to achieve this capacity must be sufficiently long to be influenced by all states of the fading process.

6.2 Capacity with Outage

Capacity with outage is used when information is transmited over the channel only when the instantaneous SNR is above a threshold value. Therefore, for certain periods of time we are in outage and receive no data. This capacity is useful when the channel is changing slowly and the channel coherence time is much greater than the symbol time. This leads to a tradeoff between maximising capacity and minimising outage probability.

6.3 Delay Limited Capacity

Delay limited capacity is associated with channel inversion when channel state information is available at the transmitter. The transmission occurs reliably at a fixed rate and is independent of the actual realisation of the fading process. However, this can be achieved only when the channel can be inverted with a finite power($E(\frac{1}{\gamma}) < \infty$). In the limit of infinite diversity, delay limited capacity equals the ergodic capacity[3].

7 Comparison between different SISO Channel Capacities

Figure 1 presents capacity results for different situations considered above for a Rayleigh fading channel and also includes the capacity in AWGN channel for comparison. The plots were generated in matlab. Capacity in an AWGN channel is greater than other capacities except for at lower SNRs where the capacity obtained using the water filling technique can actually be greater. Intuitively, this is possible since there is non-zero probability for the instantaneous SNR to be greater than the average value in a Rayleigh fading channel and more power can be put at these instances to maximise capacity. Another observation can be made about the ergodic capacity with and without transmitter CSI, both of them approach same value at high SNRs.

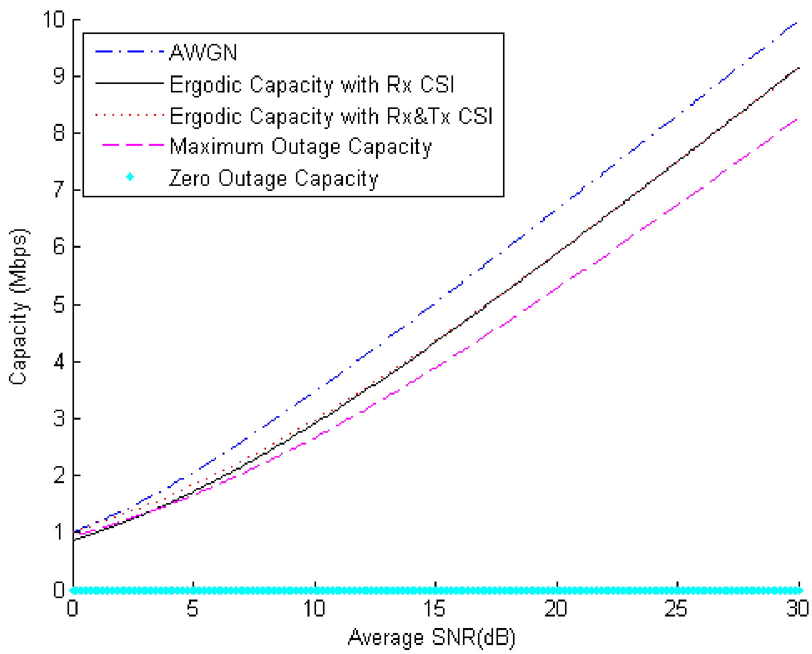

Figure 1: Comparison between different capacities in Rayleigh fading, Bandwidth B = 1Mhz

8 MIMO Overview

MIMO or Multiple Input Multiple Output communication system employs mutiple transmit and receive antennas. The purpose is to combine the signals intelligently at the transmitter and receiver to obtain diversity(lower BER) or multiplexing(higher data rates) gain or a combination of both. There are also single input multiple output(SIMO) and multiple input single output(MISO) systems and some of their capacity results will also be provided to contrast the differences. We can use the following equation to describe a MIMO system with M receive antennas and N transmit antennas :

$$y = Hx + n$$

y is M×1 received signal vector, x is N×1 transmit signal vector, H is M×N channel matrix and n is M×1 additive noise vector. The only interference present in this system is interference between the different input streams.

9 MIMO System Capacity Results

Capacity for a MIMO system can be obtained by maximising the mutual information between the channel input and output given as follows:

$$C = max_{p(x)} I(X;Y) = max_{p(x)}(H(Y) - H(Y|X))$$

H(Y) is the entropy in Y and $H(Y|X)$ is conditional entropy of Y given X. The above optimisation leads to the following expression for capacity :

$$C = max_{(Rx:Tr(Rx)=\rho)} Blog_2(det(I_M + HR_xH^H))$$

$Tr(Rx) = \rho$ comes from the total power constraint. R_x is the input covariance matrix

9.1 Static Channel known at the Transmitter and Receiver

When the channel is known at the transmitter, singular value decomposition can be performed for the matrix H resulting in the following capacity formula:

$$C = max_{\rho_i} \Sigma_i Blog_2(1 + \sigma_i^2 \rho_i)$$

Here i varies from 1 to the rank R_H of matrix H. σ_i^2 are the non-zero singular values of H. Also , maximisation must be done under the total power constraint $\Sigma_i \rho_i \leq \rho$. Water filling technique gives the optimal capacity in this case. For a SIMO or MISO system, channel matrix H is reduced to a vector h of channel gains. Capacity in this case is given by the following formula:

$$C = Blog_2(1 + \rho\|h\|^2)$$

Here ρ is the SNR $\frac{P}{\sigma^2}$ It is interesting to note than in a MIMO system capacity grows approximately linearly with the number of antennas while in a SIMO or MISO the capacity increase is logarithmic. The capacity increase in a MIMO system is not always linear and also depends on the relative placement of the antennas. If the antennas are densely packed then the different channels will be highly correlated and thus the rank of the matrix H will stay constant with increasing the number of antennas.

9.2 Static Channel known at the Receiver but unknown at the Transmitter

When the channel is unknown at the transmitter, power is uniformly distributed across all the channels resulting in the following capacity formula :

$$C = \Sigma_i Blog_2(1 + \frac{\sigma_i^2 \rho}{N})$$

The summation is over all the singular values σ_i^2 of the matrix H. N is the number of transmit antennas. For a MISO or SIMO system, capacity is given as follows :

$$C = Blog_2(1 + \frac{\rho}{N}\|h\|^2)$$

9.3 Fading Channel known at the Transmitter and Receiver

For a fading channel, channel matrix H varies with time. With CSI available at both the transmitter and receiver, channel capacity is obtained by averaging the (static channel)capacities associated with each channel realisation. This average capacity, also called the ergodic capacity is given as follows:

$$C = E_H(max_{\rho_i} \Sigma_i Blog_2(1 + \sigma_i^2 \rho_i))$$

Expectation is over all the channel realisations. Maximisation must be done under the total power constraint $\Sigma_i \rho_i \leq \rho$. However, there is an added flexibility of using a long term or short term power constraint. In the long term constraint, different powers can be allocated to different channel realisations under the constraint $E_H(P_H) \leq P_{avg}$. In the short term constraint, power associated with each channel realisation must equal the average power. Figure 2 compares the ergodic capacity of a SISO and 4×4 MIMO system with and without transmitter CSI and a bandwidth B = 1Mhz. The channel gains are modelled as zero mean circularly symmetric complex gaussians(ZMCSCG). The plot was generated using Matlab. The capacities with and without feedback approach the same value for high SNRs for both MIMO and SISO systems. Secondly, the capacity for the MIMO channel is much higher than the SISO channel

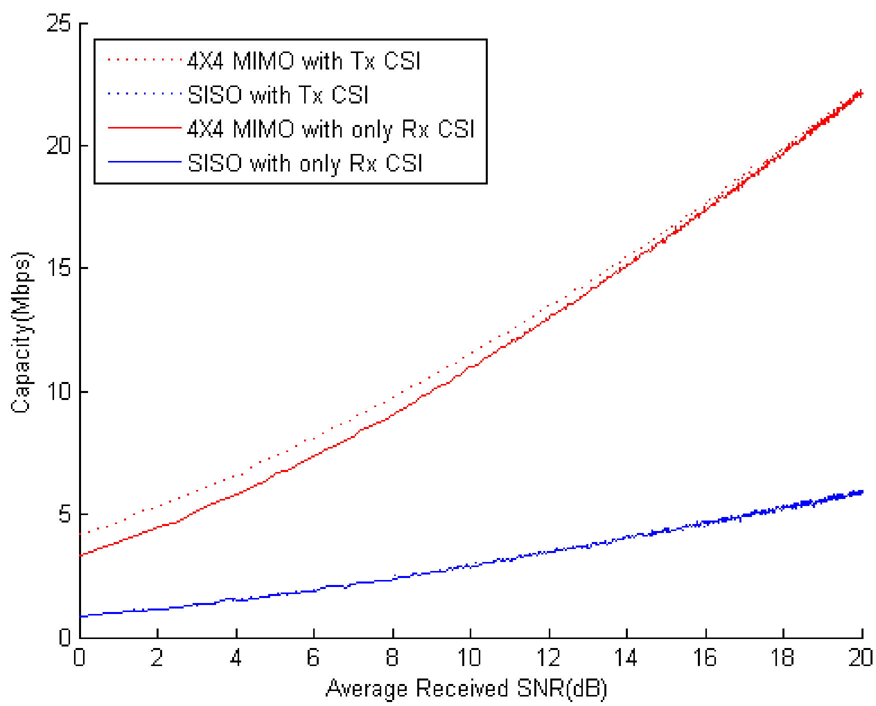

Figure 2: Ergodic Capacity with both Transmitter and Receiver CSI, Bandwidth B = 1Mhz

9.4 Fading Channel known at the Receiver but unknown at the Transmitter

When channel state is unknown at the transmitter it is best to allocate power equally over all the channels. The capacity in this case is given by the following formula:

$$C = E_H(\Sigma_i Blog_2(1 + \frac{\sigma_i^2 \rho}{N}))$$

Here N is the number of transmit antennas as stated before. Figure 2 plots this capacity.

10 Conclusion

The report presented wireless channel capacity results for single user systems in both static and fading channels. It also included the results when CSI is available and unavailable at the transmitter and receiver. Comparison between different capacities in a SISO system was made in figure 1. The report also presented capacity results for MIMO systems and compared them with SISO channel capacities in figure 2.

References

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